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# On the Periodicity Problem for Residual $r$-Fubini Sequences

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**Abstract**

For any positive integer $r$, the $r$-Fubini number with parameter $n$, denoted by $F_{n,r}$, is equal to the number of ways that the elements of a set with $n + r$ elements can be weakly ordered such that the $r$ least elements are in distinct orders. In this article we focus on the sequence of residues of the $r$-Fubini numbers modulo an arbitrary positive integer $s$ and show that this sequence is periodic and then, exhibit how to calculate its period length.

## 1 Introduction

*The Fubini numbers* (also known as the *ordered Bell numbers*) form an integer sequence in which the $n$th term counts the number of weak orderings of a set with $n$ elements. Weak ordering means that the elements can be ordered, allowing ties. Cayley [2] studied the Fubini numbers as the number of a certain kind of trees with $n+1$ terminal nodes. The Fubini numbers can also be defined as the sum of the *Stirling numbers of the second kind*, $\left\{ n \atop k \right\}$, which counts the number of partitions of an $n$-element set into $k$ non-empty subsets. The sequence of residues of the Fubini numbers modulo a positive integer $s$ was studied by Poonen [6]. He showed that this sequence is periodic and calculated the period length for each positive integer $s$.

*The $r$-Stirling numbers of the second kind* are defined as an extension to the Stirling numbers of the second kind, in which the first $r$ elements contained in distinct subsets. Similarly the $r$-Fubini numbers, which are denoted by $F_{n,r}$, are defined as the number of ways which the elements of a set with $n+r$ elements can be weakly ordered such that the first $r$ elements are in distinct places. Consider the sequence of remainders of $F_{n,r}$ modulo an arbitrary number $s \in \mathbb{N}$ in which $r$ is fixed, which is denoted by $A_{r,s}$. One can study the periodicity problem for this sequence. Mező [4] investigated this problem for $s = 10$. In this article $\omega(A_{r,s})$, the period of $A_{r,s}$, is computed for any positive integer $s$. Based on the fundamental theorem of arithmetic, $\omega(A_{r,p})$ is calculated for powers of odd primes $p^m$. The cases $s = 2^m$ are studied separately. Therefore if $s = 2^m p_1^{m_1} p_1^{m_1} \cdots p_k^{m_k}$ is the prime factorization, then the $\omega(A_{r,s})$ is equal to the least common multiple of $\omega(A_{r,p_i^{m_i}})$s and $\omega(A_{r,2^m})$, for $i = 1, 2, \ldots, k$.

Section 2 contains the basic definitions and relations. The length of the periods in the case of odd prime powers are computed in the Section 3. The similar results about the 2 powers are stated in the Section 4. The last section contains the final theorem which presents the conclusion of the article.

## 2 Basic concepts

Let $\left\{ n \atop k \right\}$ be the Stirling number of the second kind with the parameters $n$ and $k$ and let $\left\{ n \atop k \right\}_r$ be the $r$-Stirling number of the second kind with parameters $n$ and $k$. It is clear that $n \geq k \geq r$. Fubini numbers are computed as follows [4]:

$$F_n = \sum_{k=0}^{n} k! \left\{ n \atop k \right\}.$$

In a similar way we can evaluate the $r$-Fubini number $F_{n,r}$ by

$$F_{n,r} = \sum_{k=0}^{n} (k+r)! \left\{ n+r \atop k+r \right\}_r.$$

There are simple relations and formulae about $\left\{ {n \atop k} \right\}_r$ which are listed below. One can find a proof of them in [1, 4, 5] and [3, Thm. 4.5.1, p. 158].

$$\left\{ {n \atop m} \right\}_r = \left\{ {n \atop m} \right\}_{r-1} - (r-1)\left\{ {n-1 \atop m} \right\}_{r-1}, 1 \leq r \leq n \tag{1}$$

$$\left\{ {n \atop m} \right\}_1 = \left\{ {n \atop m} \right\} \tag{2}$$

$$\left\{ {n+r \atop r} \right\}_r = r^n \tag{3}$$

$$\left\{ {n+r \atop r+1} \right\}_r = (r+1)^n - r^n \tag{4}$$

$$\left\{ {n \atop m} \right\} = \frac{1}{m!}\sum_{j=1}^{m}(-1)^{m-j}\binom{m}{j}j^n \tag{5}$$

$$\left\{ {n \atop m} \right\}_r = \frac{1}{m!}\sum_{j=r}^{m}(-1)^{m-j}\binom{m}{j}j^{n-(r-1)}\left(\frac{(j-1)!}{(j-r)!}\right). \tag{6}$$

By $\varphi(n)$ we indicate the number of positive integer numbers less than $n$ and co-prime to it. It is known as Euler's totient function. The value of $\varphi(n)$ can be computed via the following relation [3, Example 4.7.3, p. 167]:

$$\varphi(n) = n\prod_{p|n}(1-\frac{1}{p}).$$

# 3 The $r$-Fubini residues modulo prime powers

Let $p$ be a prime number greater than 2 and $m$ be a positive integer. If $(F_{n,r})$ denotes the sequence of $r$-Fubini numbers for a fixed positive integer $r$, we indicate by $A_{r,q} = (F_{n,r} (\text{mod } q))$, for $n \in \mathbb{N}$, the sequence of residues of the $r$-Fubini numbers modulo the positive integer $q$. In this section we try to compute the period length of the sequence $A_{r,q}$ when $q = p^m$. This length is denoted by $\omega(A_{r,q})$.

**Proposition 1.** *Let $p$ be an odd prime and let $q = p^m$, $m \in \mathbb{N}$. If $q \leq r$, then $\omega(A_{r,q}) = 1$.*

*Proof.* The proof is very simple. Since $p \leq r$, we can deduce that $p \mid (k+r)!$, for $k \geq 0$, and by the relation $F_{n,r} = \sum_{k=0}^{n}(k+r)!\left\{ {n+r \atop k+r} \right\}_r$, we have $p \mid F_{n,r}$. Therefore $\omega(A_{r,p}) = 1$. □

As pointed out in the above proposition, it is sufficient to investigate the period length in the cases of $q > r$.

**Lemma 2.** *Let $p$ be an odd prime and $r, m \in \mathbb{N}$ with $p \geq r+1$. Then*

$$p^m - r \geq m.$$

*Proof.* For $m = 1$ the result is obvious. Suppose the inequality holds for any $m \geq 2$. Since $p(p+m) > 2(p+m) > 2p+m$, we have

$$p^2 + pm - p \geq p + m. \tag{7}$$

Since $p - 1 \geq r$, the induction hypothesis can be reformulated to $p^m \geq p - 1 + m$. Multiplication by $p$ results $p^{m+1} \geq p^2 + pm - p$. By (7) we have $p^{m+1} \geq p + (m+1) - 1$. □

**Theorem 3.** *Let $p$ be an odd prime and $q = p^m$. After the $(m-1)$th term the sequence $A_{r,q}$ has a period with length $\omega(A_{r,q}) = \varphi(q)$. In other words, $F_{n+\varphi(q),r} \equiv F_{n,r} \pmod q$, for $n \geq m - 1$.*

*Proof.* If $n \geq q - r - 1$ we can write

$$\begin{aligned}
F_{n+\varphi(q),r} - F_{n,r} &= \sum_{k=0}^{n+\varphi(q)} (k+r)! \left\{ {n+\varphi(q)+r \atop k+r} \right\}_r - \sum_{k=0}^{n} (k+r)! \left\{ {n+r \atop k+r} \right\}_r \\
&\equiv \sum_{k=0}^{q-r-1} (k+r)! \left( \left\{ {n+\varphi(q)+r \atop k+r} \right\}_r - \left\{ {n+r \atop k+r} \right\}_r \right) \\
&\equiv \sum_{k=0}^{q-r-1} \sum_{j=r}^{k+r} (-1)^{k+r-j} \binom{k+r}{j} j^{n+1} \left( \frac{(j-1)!}{(j-r)!} \right) (j^{\varphi(q)} - 1) \pmod q.
\end{aligned}$$

If $j = cp$, $c \in \mathbb{N}$, then $j^{n+1} = (cp)^{q-r+h}$, for some $h \geq 0$, so from Lemma 2 it follows that $j^{n+1} \equiv 0 \pmod q$. If $\gcd(j, q) = 1$, by Euler's theorem $j^{\varphi(q)} - 1 \equiv 0 \pmod q$, so the right hand side of the above congruence relation vanished and we have

$$F_{n+\varphi(q),r} \equiv F_{n,r} \pmod q, \text{ for } n \geq q - r - 1. \tag{8}$$

If $m - 1 \leq n < q - r - 1$ then

$$\begin{aligned}
F_{n+\varphi(q),r} - F_{n,r} &\equiv \sum_{k=0}^{q-r-1} (k+r)! \left( \left\{ {n+\varphi(q)+r \atop k+r} \right\}_r - \left\{ {n+r \atop k+r} \right\}_r \right) \\
&\quad - \sum_{k=n+\varphi(q)+1}^{q-r-1} (k+r)! \left\{ {n+\varphi(q)+r \atop k+r} \right\}_r + \sum_{k=n+1}^{q-r-1} (k+r)! \left\{ {n+r \atop k+r} \right\}_r \\
&\equiv \sum_{k=0}^{q-r-1} \sum_{j=r}^{k+r} (-1)^{k+r-j} \binom{k+r}{j} j^{n+1} \left( \frac{(j-1)!}{(j-r)!} \right) (j^{\varphi(q)} - 1) \\
&\quad - \sum_{k=n+\varphi(q)+1}^{q-r-1} (k+r)! \left\{ {n+\varphi(q)+r \atop k+r} \right\}_r + \sum_{k=n+1}^{q-r-1} (k+r)! \left\{ {n+r \atop k+r} \right\}_r \pmod q.
\end{aligned}$$

Since $n \geq m-1$, in the indices where $j = cp$, $c \in \mathbb{N}$, we have $j^{n+1} = (cp)^{m+h}$, for some $h \geq 0$, and it is deduced that $j^{n+1} \equiv 0 \pmod{q}$. When $\gcd(j,q) = 1$, again $j^{\varphi(q)} - 1 \equiv 0 \pmod{q}$ by Euler's theorem. In the sums $\sum_{k=n+1}^{q-r-1}(k+r)!\left\{ {n+r \atop k+r} \right\}_r$ and $\sum_{k=n+\varphi(q)+1}^{q-r-1}(k+r)!\left\{ {n+\varphi(q)+r \atop k+r} \right\}_r$ the upper parameter of the $r$-Stirling number is less than the lower one, and therefore these two sums are equal to zero. So

$$F_{n+\varphi(q),r} - F_{n,r} \equiv \sum_{k=0}^{q-r-1} \sum_{j=r}^{k+r} (-1)^{k+r-j} \binom{k+r}{j} j^{n+1} \left( \frac{(j-1)!}{(j-r)!} \right) (j^{\varphi(q)} - 1) \equiv 0 \pmod{q},$$

and therefore

$$F_{n+\varphi(q),r} \equiv F_{n,r} \pmod{q} \text{ for } m-1 \leq n < q-r-1. \tag{9}$$

Combining results (8) and (9) gives $F_{n+\varphi(q),r} \equiv F_{n,r} \pmod{q}$, for $n \geq m-1$. □

# 4 The $r$-Fubini residues modulo powers of 2

As in many other computations in number theory, the case of $p = 2$ has its own difficulties that require special attention. In the case of powers of 2, initially we calculate the residues of 2-Fubini numbers and then use the results in the case of the $r$-Fubini numbers. We classify the sequence of remainders of 2-Fubini numbers modulo $2^m$, $m \geq 7$, in Theorem 6 and then, work on remainders of the $r$-Fubini numbers modulo $2^m$, $m \geq 7$ in Theorem 9. The special cases will be proved in Theorems 4, 7 and 8. The trivial cases in which $2^m \leq r$ with period length 1 are omitted.

**Theorem 4.** *If $3 \leq m \leq 6$, then after the $(m-1)$th term the sequence $A_{2,2^m}$ has a period with length $\omega(A_{2,2^m}) = 2$.*

*Proof.* By using the formula $F_{n,2} = \sum_{k=0}^{n}(k+2)!\left\{ {n+2 \atop k+2} \right\}_2$ we prove that $F_{n+2,2} - F_{n,2} \equiv 0 \pmod{64}$. Then $F_{n+2,2} - F_{n,2} \equiv 0 \pmod{2^m}$ for $3 \leq m \leq 5$.

$$\begin{aligned} F_{n+2,2} - F_{n,2} &= \sum_{k=0}^{n+2}(k+2)!\left\{ {n+4 \atop k+2} \right\}_2 - \sum_{k=0}^{n}(k+2)!\left\{ {n+2 \atop k+2} \right\}_2 \\ &\equiv \sum_{k=0}^{5}(k+2)! \left( \left\{ {n+4 \atop k+2} \right\}_2 - \left\{ {n+2 \atop k+2} \right\}_2 \right) \\ &\equiv \sum_{k=0}^{5} \sum_{j=2}^{k+2} (-1)^{k+2-j} \binom{k+2}{j} j^{n+1}(j^2-1)(j-1) \pmod{64}. \end{aligned}$$

In the case $m = 6$ then $n \geq 5$, so if $j$ is even, then $j^{n+1} = (2c)^{6+h}$, for some $h \geq 0$ and therefore $64 \mid j^{n+1}$. For odd $j$ we have $\gcd(j, 64) = 1$, so by Euler's theorem we have

$j^{32} \equiv 1 \pmod{64}$, and therefore $j^{n+1+32} \equiv j^{n+1} \pmod{64}$. This implies that

$$F_{n+2,2} - F_{n,2} \equiv \sum_{k=0}^{5} \sum_{l=1}^{\lfloor (k+1)/2 \rfloor} (-1)^{k+2-(2l+1)} \binom{k+2}{2l+1} (2l+1)^{n+1} \left((2l+1)^2 - 1\right) \times 2l$$
$$\equiv 16 \sum_{k=0}^{5} (-1)^{k+1} \sum_{l=1}^{\lfloor (k+1)/2 \rfloor} \binom{k+2}{2l+1} (2l+1)^{n+1} \left(\frac{l(l+1)}{2}\right) l \pmod{64}.$$

Enumerating the last summation for $2 \leq n \leq 33$ shows that it is divisible by 64 and because of periodicity of remainders of $j^{n+1}$ modulo 64, the result follows. □

Analogous to Lemma 2, it can be easily deduced by induction, showing that for each positive integer $m > 1$ we have

$$2^m - 2 \geq m \; . \tag{10}$$

This can be shown by using the relation $2^{m+1} \geq 2m + 4 > m + 3$, for $m > 1$. The following lemma provides a simple but essential relation used in the next theorem. Its proof is provided in Appendix A.

**Lemma 5.** *For $m \geq 7$ and $5 \leq i \leq 2^{m-6}$ we have $2^{m-6} - i \mid 2^{i-5}\binom{2^{m-6}-1}{i}$.*

**Theorem 6.** *If $m \geq 7$, after the $(m-1)$th term, the sequence $A_{2,2^m}$ has a period with length $\omega(A_{2,2^m}) = 2^{m-6}$.*

*Proof.* In the case of $n \geq 2^m - 3$, from (10) we can deduce that $n \geq 2^m - 3 \geq m - 1$. So we have

$$F_{n+2^{m-6},2} - F_{n,2} \equiv \sum_{k=0}^{n+2^{m-6}} (k+2)! \left\{ {n+2^{m-6}+2 \atop k+2} \right\}_2 - \sum_{k=0}^{n} (k+2)! \left\{ {n+2 \atop k+2} \right\}_2$$
$$\equiv \sum_{k=0}^{2^m-3} (k+2)! \left( \left\{ {n+2^{m-6}+2 \atop k+2} \right\}_2 - \left\{ {n+2 \atop k+2} \right\}_2 \right)$$
$$\equiv \sum_{k=0}^{2^m-3} \sum_{j=2}^{k+2} (-1)^{k+2-j} \binom{k+2}{j} j^{n+1} (j^{2^{m-6}} - 1)(j-1) \pmod{2^m}.$$

When $j$ is even, then $j^{n+1} = (2c)^{2^m-2+h}$, for some $h \geq 0$. So by (10), $2^m \mid j^{n+1}$. For odd $j$ we have

$$F_{n+2^{m-6},2} - F_{n,2} \equiv \sum_{k=0}^{2^m-3} \sum_{l=1}^{\lfloor (k+1)/2 \rfloor} (-1)^{k+2-(2l+1)} \binom{k+2}{2l+1} (2l+1)^{n+1}((2l+1)^{2^{m-6}} - 1) \times 2l$$

$$\equiv 2^{m-4} \sum_{k=0}^{2^m-3} (-1)^{k+1} \sum_{l=1}^{\lfloor (k+1)/2 \rfloor} \binom{k+2}{2l+1} (2l+1)^{n+1} \left( \frac{(2l+1)^{2^{m-6}} - 1}{2^{m-5}} \right) l$$

$$\equiv 2^{m-4} \sum_{k=0}^{2^m-3} (-1)^{k+1} \sum_{l=1}^{\lfloor (k+1)/2 \rfloor} \binom{k+2}{2l+1} (2l+1)^{n+1} \sum_{i=1}^{2^{m-6}} l^i 2^{i-1} \left( \frac{(2^{m-6}-1)!}{i!(2^{m-6}-i)!} \right)$$

$$\times l \pmod{2^m}.$$

The last expression contains $m-4$ factors of 2, so it is sufficient to prove that the last summation is divisible by 16. This summation is denoted by $\mathcal{S}$. Simplify the summation $\sum_{i=1}^{2^{m-6}} l^i 2^{i-1} \frac{(2^{m-6}-1)!}{i!(2^{m-6}-i)!}$ and using Lemma 5 gives

$$\sum_{i=1}^{2^{m-6}} l^i 2^{i-1} \left( \frac{(2^{m-6}-1)!}{i!(2^{m-6}-i)!} \right) \equiv \sum_{i=1}^{4} l^i 2^{i-1} \left( \frac{(2^{m-6}-1)!}{i!(2^{m-6}-i)!} \right) \equiv l + l^2(2^{m-6}-1)$$

$$+ \frac{l^3 \times 2(2^{m-6}-1)(2^{m-6}-2)}{3} + \frac{l^4(2^{m-6}-1)(2^{m-6}-2)(2^{m-6}-3)}{3} \pmod{16}.$$

Assume $m \geq 10$ (the case $7 \leq m \leq 9$ is studied at the end of the proof). So $16 \mid 2^{m-6}$. Let $3a = 2(2^{m-6}-1)(2^{m-6}-2)$ and $3b = (2^{m-6}-1)(2^{m-6}-2)(2^{m-6}-3)$. Then $3a \equiv 4 \pmod{16}$ and $3b \equiv -6 \pmod{16}$. Therefore $a \equiv -4 \pmod{16}$ and $b \equiv -2 \pmod{16}$. So the proof continues as follows:

$$\mathcal{S} \equiv \sum_{k=0}^{2^m-3} (-1)^{k+1} \left( \sum_{l=1}^{\lfloor (k+1)/2 \rfloor} \binom{k+2}{2l+1} (2l+1)^{n+1} (l - l^2 - 4l^3 - 2l^4) l \right) \pmod{16}$$

$$\mathcal{S} \equiv \sum_{k=0}^{2^m-3} (-1)^{k+1} \sum_{l=1}^{\lfloor (k+1)/2 \rfloor} \binom{k+2}{2l+1} (2l+1)^{n+1} \left( \frac{l(l+1)}{2} \right) (-2l^2 - 2l + 1) l \pmod{8}.$$

Let $P(l)$ and $A(k,r,n)$ be the remainder of $\frac{1}{2}(2l+1)^{n+1}(l(l+1))(-2l^2-2l+1)l$ and $\sum_{l=-\infty}^{\infty} \binom{k+2}{2l+r} P(l)$ divided by 8, respectively. By Pascal's identity, we have $\binom{k+2}{2l+r} = \binom{k+1}{2l+r} + \binom{k+1}{2l+r-1}$ and therefore

$$\sum_{l=-\infty}^{\infty} \binom{k+2}{2l+r} P(l) = \sum_{l=-\infty}^{\infty} \binom{k+1}{2l+r} P(l) + \sum_{l=-\infty}^{\infty} \binom{k+1}{2l+r-1} P(l),$$

so

$$A(k,r,n) = A(k-1,r,n) + A(k-1,r-1,n). \tag{11}$$

We can write

$$A(k, r+32, n) \equiv \sum_{l=-\infty}^{\infty} \binom{k+2}{2l+r+32} P(l) \pmod 8.$$

The sequence $(P(l))_{l=-\infty}^{\infty}$ has period 16, so $P(l+16) = P(l)$. Set $l' = l + 16$, then

$$A(k, r+32, n) \equiv \sum_{l'=-\infty}^{\infty} \binom{k+2}{2l'+r} P(l') \equiv A(k, r, n) \pmod 8. \tag{12}$$

Since $\gcd(2l+1, 16) = 1$, Euler's theorem implies $(2l+1)^8 \equiv 1 \pmod{16}$ and therefore $(2l+1)^{n+1+8} \equiv (2l+1)^{n+1} \pmod{16}$. The quantity $A(6, r, n)$ vanishes for $1 \le r \le 32$ and $9 \le n \le 24$, by enumeration, then by (11) and (12), we deduce that

$$A(k, r, n) = 0, \text{ for } k \ge 6. \tag{13}$$

Therefore

$$\begin{aligned} A(k, 1, n) &\equiv \sum_{l=-\infty}^{\infty} \binom{k+2}{2l+1} (2l+1)^{n+1} \left(\frac{l(l+1)}{2}\right) (-2l^2 - 2l + 1) l \\ &\equiv \sum_{l=1}^{\lfloor (k+1)/2 \rfloor} \binom{k+2}{2l+1} (2l+1)^{n+1} \left(\frac{l(l+1)}{2}\right) (-2l^2 - 2l + 1) l \equiv 0 \pmod 8, \end{aligned}$$

for $k \ge 6$. If $1 \le k \le 5$, $9 \le n \le 24$ and $1 \le r \le 32$ we have $\sum_{k=1}^{5} (-1)^{k+1} A(k, r, n) \equiv 0 \pmod 8$. The period length of $A(k, r, n)$ with respect to $r$ and $n$ implies that

$$\sum_{k=1}^{5} (-1)^{k+1} A(k, 1, n) \equiv 0 \pmod 8, \text{ for } n \ge 9.$$

Combining this with (13) we have

$$\mathcal{S} \equiv \sum_{k=1}^{2^m - 3} (-1)^{k+1} A(k, 1, n) \equiv 0 \pmod 8, \text{ for } n \ge 0.$$

So the result follows in the case of $n \geq 2^m - 3$. If $m-1 \leq n < 2^m - 3$ we can write

$$\begin{aligned} F_{n+2^{m-6},2} - F_{n,2} &= \sum_{k=0}^{n+2^{m-6}} (k+2)! \left\{ {n+2^{m-6}+2 \atop k+2} \right\}_2 - \sum_{k=0}^{n} (k+2)! \left\{ {n+2 \atop k+2} \right\}_2 \\ &= \sum_{k=0}^{2^m-3} (k+2)! \left( \left\{ {n+2^{m-6}+2 \atop k+2} \right\}_2 - \left\{ {n+2 \atop k+2} \right\}_2 \right) \\ &- \sum_{k=n+2^{m-6}+1}^{2^m-3} (k+2)! \left\{ {n+2^{m-6}+2 \atop k+2} \right\}_2 + \sum_{k=n+1}^{2^m-3} (k+2)! \left\{ {n+2 \atop k+2} \right\}_2 \\ &\equiv \sum_{k=0}^{2^m-3} \sum_{j=1}^{k+2} (-1)^{k+2-j} \binom{k+2}{j} j^{n+1} (j^{2^{m-6}} - 1)(j-1) \pmod{2^m}. \end{aligned}$$

When $j$ is even, then $j^{n+1} = (2c)^{m+h}$, for some $h \geq 0$, so $2^m \mid j^{n+1}$. Since $m \geq 10$, for odd $j$ we have

$$\begin{aligned} &\sum_{k=0}^{2^m-3} \sum_{j=1}^{k+2} (-1)^{k+2-j} \binom{k+2}{j} j^{n+1} (j^{2^{m-6}} - 1)(j-1) \\ &\equiv 2^{m-4} \sum_{k=0}^{2^m-3} (-1)^{k+1} \sum_{l=1}^{\lfloor (k+1)/2 \rfloor} \binom{k+2}{2l+1} (2l+1)^{n+1} (l - l^2 - 4l^3 - 2l^4) l \pmod{2^m}. \end{aligned}$$

The last summation is exactly the $\mathcal{S}$ and the proof will be similar as above. Combine with the previous case we have the following congruence relation

$$F_{n+2^{m-6},2} \equiv F_{n,2} \pmod{2^m}, \text{ for } m \geq 10. \tag{14}$$

In the case where $7 \leq m \leq 9$, the remainder value of the sum

$$\sum_{k=0}^{2^m-3} (-1)^{k+1} \sum_{l=1}^{\lfloor (k+1)/2 \rfloor} \binom{k+2}{2l+1} (2l+1)^{n+1} \left( \sum_{i=1}^{4} l^i 2^{i-1} \frac{(2^{m-6}-1)!}{i!(2^{m-6}-i)!} \right) l$$

modulo 16 is computed for $m-1 \leq n \leq m+14$. Divisibility of all these values by 16 implies that the recent sum is divisible by 16, and therefore

$$F_{n+2^{m-6},2} \equiv F_{n,2} \pmod{2^m}, \text{ for } 7 \leq m \leq 9. \tag{15}$$

Summing up the congruence relations (14) and (15) gives

$$\omega(A_{2,2^m}) = 2^{m-6}, \text{ for } m \geq 7.$$

□

**Theorem 7.** *For $m = 1$ and $m = 2$, the sequence $A_{r,2^m}$ is periodic from the first term and the period length is $\omega(A_{r,2^m}) = 1$.*

*Proof.* The proof of this theorem is divided into three cases. For $r = 2$ we have

$$\begin{aligned} F_{n+1,2} - F_{n,2} &= \sum_{k=0}^{n+1}(k+2)!\left\{{n+3 \atop k+2}\right\}_2 - \sum_{k=0}^{n}(k+2)!\left\{{n+2 \atop k+2}\right\}_2 \\ &\equiv 2\left(\left\{{n+3 \atop 2}\right\}_2 - \left\{{n+2 \atop 2}\right\}_2\right) + 6\left(\left\{{n+3 \atop 3}\right\}_2 - \left\{{n+2 \atop 3}\right\}_2\right) \pmod 4 \\ &= 2\left(2^{n+1} - 2^n\right) + 6\left(3^{n+1} - 2^{n+1} - (3^n - 2^n)\right) \\ &= 2^{n+1} + 6(2 \times 3^n - 2^n) = 4(2^{n-1} + 3^{n+1} - 3 \times 2^{n-1}) \\ &= 4(3^{n+1} - 2^n) \equiv 0 \pmod 4. \end{aligned}$$

So we can deduce that $\omega(A_{2,4}) = 1$ and obviously $\omega(A_{2,2}) = 1$.

For $r = 3$ we can write

$$\begin{aligned} F_{n+1,3} - F_{n,3} &= \sum_{k=0}^{n+1}(k+3)!\left\{{n+4 \atop k+3}\right\}_3 - \sum_{k=0}^{n}(k+3)!\left\{{n+3 \atop k+3}\right\}_3 \\ &\equiv 6\left(\left\{{n+4 \atop 3}\right\}_3 - \left\{{n+3 \atop 3}\right\}_3\right) \pmod 4 \\ &= 6\left(3^{n+1} - 3^n\right) = 6 \times 2 \times 3^n = 4 \times 3^{n+1} \equiv 0 \pmod 4. \end{aligned}$$

Therefore we have $\omega(A_{3,4}) = 1$ and $\omega(A_{3,2}) = 1$.

Finally if $r \geq 4$, let $r = 4 + h$, for some $h \geq 0$, then

$$F_{n+1,r} - F_{n,r} = \sum_{k=0}^{n+1}(k+r)!\left\{{n+1+r \atop k+r}\right\}_r - \sum_{k=0}^{n}(k+r)!\left\{{n+r \atop k+r}\right\}_r.$$

Since $4 \mid (k+r)!$, for all $k \geq 0$, we can write $F_{n+1,r} - F_{n,r} \equiv 0 \pmod 4$. Therefore $\omega(A_{r,4}) = 1$ and $\omega(A_{r,2}) = 1$. □

**Theorem 8.** *If $3 \leq m \leq 6$, after the $(m-1)$th term, the sequence $A_{r,2^m}$ has a period with length $\omega(A_{r,2^m}) = 2$.*

*Proof.* The proof of this theorem is similar to the proof of Theorem 4. It is enough to prove the theorem for $m = 6$; then the result follows for $m = 3, 4$ and $5$. Since $n \geq m - 1$, then for $m=6$ we have $n\geq 5$. For $3\leq r\leq 7$ we have

$$\begin{aligned}
F_{n+2,r}-F_{n,r} &= \sum_{k=0}^{n+2}(k+r)!\left\{{n+2+r \atop k+r}\right\}_r-\sum_{k=0}^{n}(k+r)!\left\{{n+r \atop k+r}\right\}_r \\
&\equiv \sum_{k=0}^{7-r}\sum_{j=r}^{k+r}(-1)^{k+r-j}\binom{k+r}{j}j^{n+1}(j^2-1)\left(\frac{(j-1)!}{(j-r)!}\right) \\
&\equiv \sum_{k=0}^{7-r}\sum_{j=r}^{k+r}(-1)^{k+r-j}\binom{k+r}{j}j^{n+1}(j^2-1)(j-1)\left(\frac{(j-2)!}{(j-r)!}\right) \pmod{64}.
\end{aligned}$$

When $j$ is even, then $j^{n+1}=(2c)^{6+h}$, for some $h\geq 0$, and so $64\mid j^{n+1}$. For odd $j$ we have $\gcd(j,64)=1$ and Euler's theorem gives $j^{32}\equiv 1 \pmod{64}$. Therefore $j^{n+1+32}\equiv j^{n+1} \pmod{64}$, and we can write

$$\begin{aligned}
&\sum_{k=0}^{7-r}\sum_{j=r}^{k+r}(-1)^{k+r-j}\binom{k+r}{j}j^{n+1}(j^2-1)(j-1)\frac{(j-2)!}{(j-r)!} \\
&\equiv \sum_{k=0}^{7-r}\sum_{l=\lfloor r/2\rfloor}^{\lfloor (k+r-1)/2\rfloor}(-1)^{k+r-(2l+1)}\binom{k+r}{2l+1}(2l+1)^{n+1}((2l+1)^2-1)\times 2l\left(\frac{(2l-1)!}{(2l+1-r)!}\right) \\
&\equiv 16\sum_{k=0}^{7-r}(-1)^{k+r-1}\sum_{l=\lfloor r/2\rfloor}^{\lfloor (k+r-1)/2\rfloor}\binom{k+r}{2l+1}(2l+1)^{n+1}\left(\frac{l(l+1)}{2}\right)l\left(\frac{(2l-1)!}{(2l+1-r)!}\right) \pmod{64}.
\end{aligned}$$

By computation we see that the recent summation is divisible by 4, for $2\leq n\leq 33$. So the proof for $3\leq r\leq 7$ is completed.

If $r\geq 8$, since $64\mid 8!$, then $64\mid (k+r)!$, and

$$F_{n+2,r}-F_{n,r}=\sum_{k=0}^{n+2}(k+r)!\left\{{n+2+r \atop k+r}\right\}_r-\sum_{k=0}^{n}(k+r)!\left\{{n+r \atop k+r}\right\}_r\equiv 0 \pmod{64},$$

so $\omega(A_{r,2^6})=2$, for $r\geq 8$, and the proof is completed. □

**Theorem 9.** *If $m\geq 7$, after the $(m-1)$th term, the sequence $A_{r,2^m}$ has a period with length $\omega(A_{r,2^m})=2^{m-6}$.*

*Proof.* The proof of this theorem is similar to the proof of Theorem 6. In the case of

$n \geq 2^m - r - 1$ and $r \geq 8$ we have

$$\begin{aligned} F_{n+2^{m-6},r} - F_{n,r} &= \sum_{k=0}^{n+2^{m-6}} (k+r)! \left\{ \begin{matrix} n+2^{m-6}+r \\ k+r \end{matrix} \right\}_r - \sum_{k=0}^{n} (k+r)! \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r \\ &\equiv \sum_{k=0}^{2^m-r-1} (k+r)! \left( \left\{ \begin{matrix} n+2^{m-6}+r \\ k+r \end{matrix} \right\}_r - \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r \right) \\ &\equiv \sum_{k=0}^{2^m-r-1} \sum_{j=r}^{k+r} (-1)^{k+r-j} \binom{k+r}{j} j^{n+1} (j^{2^{m-6}} - 1) \left( \frac{(j-1)!}{(j-r)!} \right) \pmod{2^m}. \end{aligned}$$

In the case of $2^m > r > 2^m - m$, since $m \geq 7$ this implies that $r > 2^m - m \geq 2^{m-1}$, so

$$2^m \mid (2^{m-1})! \mid (k+r)!, \text{ for each } k \geq 0.$$

Therefore both summations in the above first equation are zero modulo $2^m$ and in this case $\omega(A_{r,2^m}) = 2^{m-6}$. When $r \leq 2^m - m$, if $j$ is even then $j^{n+1} = (2c)^{2^m-r+h}$, for some $h \geq 0$. So $2^m \mid j^{n+1}$. For odd $j$ we have $(j, 2^{m-5}) = 1$, and $2^{m-5} \mid j^{2^{m-6}} - 1$ by Euler's theorem. Since $r \geq 8$ we can write $\frac{(j-1)!}{(j-r)!} = \left( \frac{(j-8)!}{(j-r)!} \right) \prod_{i=1}^{7} (j-i)$. Therefore $32 \mid \frac{(j-1)!}{(j-r)!}$ and $2^m \mid (j^{2^{m-6}} - 1) \left( \frac{(j-1)!}{(j-r)!} \right)$.

In the case of $m - 1 \leq n < 2^m - r - 1$ and $r \geq 8$ we have

$$\begin{aligned} F_{n+2^{m-6},r} - F_{n,r} &= \sum_{k=0}^{n+2^{m-6}} (k+r)! \left\{ \begin{matrix} n+2^{m-6}+r \\ k+r \end{matrix} \right\}_r - \sum_{k=0}^{n} (k+r)! \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r \\ &\equiv \sum_{k=0}^{2^m-r-1} (k+r)! \left( \left\{ \begin{matrix} n+2^{m-6}+r \\ k+r \end{matrix} \right\}_r - \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r \right) \\ &\quad - \sum_{k=n+2^{m-6}+1}^{2^m-r-1} (k+r)! \left\{ \begin{matrix} n+2^{m-6}+r \\ k+r \end{matrix} \right\} + \sum_{k=n+1}^{2^m-r-1} (k+r)! \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\} \pmod{2^m} \\ &= \sum_{k=0}^{2^m-r-1} (k+r)! \left( \left\{ \begin{matrix} n+2^{m-6}+r \\ k+r \end{matrix} \right\}_r - \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r \right) + 0, \end{aligned}$$

and the proof proceeds as in the previous case. In the case of $3 \leq r \leq 7$ one can deduce similarly to the proof of Theorem 6 that

$$F_{n+2^{m-6},r} - F_{n,r} \equiv \sum_{k=0}^{2^m-r-1} \sum_{j=r}^{k+r} (-1)^{k+r-j} \binom{k+r}{j} j^{n+1} (j^{2^{m-6}} - 1) \left( \frac{(j-1)!}{(j-r)!} \right) \pmod{2^m}.$$

Exactly the same as Theorem 6, the terms with even $j$ vanish and only the terms with odd

$j$ remain. So we have

$$\begin{aligned}
F_{n+2^{m-6},r} - F_{n,r} &\equiv \sum_{k=0}^{2^m-r-1} \sum_{l=\lfloor r/2 \rfloor}^{\lfloor (k+r-1)/2 \rfloor} (-1)^{k+r-(2l+1)} \binom{k+r}{2l+1} (2l+1)^{n+1}((2l+1)^{2^{m-6}}-1) \\
&\quad \times \left( \frac{((2l+1)-1)!}{((2l+1)-r)!} \right) \\
&\equiv 2^{m-5} \sum_{k=0}^{2^m-r-1} (-1)^{k+r-1} \sum_{l=\lfloor r/2 \rfloor}^{\lfloor (k+r-1)/2 \rfloor} \binom{k+r}{2l+1} (2l+1)^{n+1} \\
&\quad \times \left( \sum_{i=1}^{2^{m-6}} l^i 2^{i-1} \left( \frac{(2^{m-6}-1)!}{i!(2^{m-6}-i)!} \right) \right) \left( \frac{(2l)!}{(2l-r+1)!} \right) \pmod{2^m}.
\end{aligned}$$

Since $\gcd(2l+1, 16) = 1$, Euler's theorem shows that $(2l+1)^{n+1+8} \equiv (2l+1)^{n+1} \pmod{16}$. If $m \geq 10$, we have

$$\begin{aligned}
F_{n+2^{m-6},r} - F_{n,r} &\equiv 2^{m-4} \sum_{k=0}^{2^m-r-1} (-1)^{k+r-1} \sum_{l=\lfloor r/2 \rfloor}^{\lfloor (k+r-1)/2 \rfloor} \binom{k+r}{2l+1} (2l+1)^{n+1} \left( \frac{l(l+1)}{2} \right) \\
&\quad \times (-2l^2 - 2l + 1) \left( \frac{(2l)!}{(2l-r+1)!} \right) \pmod{2^m}.
\end{aligned}$$

Therefore it is sufficient to compute the above summation (without factor $2^{m-4}$) for $3 \leq r \leq 7$ and $9 \leq n \leq 16$ to show that it is divisible by 16.

For $7 \leq m \leq 9$ we evaluate the sum

$$\sum_{k=0}^{2^m-r-1} (-1)^{k+r-1} \sum_{l=\lfloor r/2 \rfloor}^{\lfloor (k+r-1)/2 \rfloor} \binom{k+r}{2l+1} (2l+1)^{n+1} \sum_{i=1}^{2^{m-6}} l^i 2^{i-1} \frac{(2^{m-6}-1)!}{i!(2^{m-6}-i)!} \left( \frac{(2l)!}{(2l-r+1)!} \right)$$

for $m-1 \leq n \leq m+6$ to show that it is divisible by 32. Then it follows that $\omega(A_{r,2^m}) = 2^{m-6}$, for all $m \geq 7$. □

# 5 The conclusion

We now state the final theorem, which shows how to compute $\omega(A_{r,s})$ for any $s \in \mathbb{N}$.

**Theorem 10.** *Let $s \in \mathbb{N}$ and $s > 1$ with the prime factorization $s = 2^m p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ and let $D = \{p_i^{m_i} \mid p_i^{m_i} > r, 1 \leq i \leq k\}$. Define $E = \{m_i - 1 \mid p_i^{m_i} \in D\}$, $F = \{\varphi(p_i^{m_i}) \mid p_i^{m_i} \in D\}$ and $a = \max(E \cup \{m-1\})$ and let $b$ be the least common multiple* (lcm) *of the elements of $F$. Then*

$$\omega(A_{r,s}) = \begin{cases} b, & \text{if } 0 \leq m \leq 2 \text{ or } 2^m \leq r; \\ \mathrm{lcm}(2, b), & \text{if } 3 \leq m \leq 6 \text{ and } 2^m > r \ ; \\ \mathrm{lcm}(2^{m-6}, b), & \text{if } m \geq 7 \text{ and } 2^m > r, \end{cases} \tag{16}$$

*and periodicity of the sequence $A_{r,s}$ is seen after the $a$-th term.*

*Proof.* Let $l$ be the right hand side of (16). For each $d \in D \cup \{2^m\}$, $\omega(A_{r,d}) \mid l$ and for each $p_j^{m_j} \notin D$ such that $1 \leq j \leq k$, we have $1 = \omega(A_{r,p_j^{m_j}}) \mid l$, so

$$\begin{aligned} F_{n+l,r} &\equiv F_{n,r} \pmod{2^m} \\ F_{n+l,r} &\equiv F_{n,r} \pmod{p_i^{m_i}}, \text{ for } i = 1, 2, \ldots, k. \end{aligned}$$

Since $\gcd(2^m, p_1^{m_1}, p_2^{m_2}, \ldots, p_k^{m_k}) = 1$, the multiplication of all above congruence relations gives the required result. □

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# A Proof of Lemma 5

After simplifying the lemma's relation we have

$$\frac{2^{i-5}\binom{2^{m-6}-1}{i}}{2^{m-6}-i} = \frac{2^{i-5}(2^{m-6}-1)(2^{m-6}-2)\cdots(2^{m-6}-i+1)}{i!}. \tag{17}$$

It is sufficient to show that the right hand side of (17) is integer. We know that $\binom{2^{m-6}}{i} \in \mathbb{N}$, i.e.,

$$i! \mid 2^{m-6}(2^{m-6}-1)\cdots(2^{m-6}-i+1).$$

If $O_i$ denotes the product of the odd factors of $i!$, since $(O_i, 2^{m-6}) = 1$, then $O_i \mid (2^{m-6} - 1)\cdots(2^{m-6}-i+1)$. So in (17) we only need to prove that

$$\nu_2(2^{i-5}(2^{m-6}-1)(2^{m-6}-2)\cdots(2^{m-6}-i+1)) \geq \nu_2(i!),$$

where by $\nu_2(x)$ we mean that $2^{\nu_2(x)} \mid x$, but $2^{\nu_2(x)+1} \nmid x$. Let $A = \nu_2((2^{m-6}-1)(2^{m-6}-2)\cdots(2^{m-6}-i+1))$ and $B = \nu_2(i!)$. Let $e$ be the unique integer such that $2^e \leq i < 2^{e+1}$. So

$$A = \sum_{k=1}^{e} \lfloor \frac{i-1}{2^k} \rfloor, \quad B = \sum_{k=1}^{e} \lfloor \frac{i}{2^k} \rfloor. \tag{18}$$

If we show that

$$B - A \leq e \tag{19}$$

then the lemma is concluded if it is proved that

$$i + A \geq B + 5. \tag{20}$$

It can easily be shown that $B = \nu_2(i!)$ and $A = \nu_2((i-1)!)$, so $B - A = \nu_2(i)$. Since $2^e \leq i < 2^{e+1}$, therefore $\nu_2(i) \leq e$ and (19) follows. For $e = 2$, integer possibilities for inequality (20) are as follows:

| $i$ | $A$ | $B$ |
|---|---|---|
| 5 | 3 | 3 |
| 6 | 3 | 4 |
| 7 | 4 | 4 |

For $e \geq 3$ one can deduce by simple induction that

$$2^e \geq e + 5,$$

so $i \geq 2^e \geq e + 5$. Add $B - e$ to these inequalities and use (19) demonstrates (20) for $i \geq 8$.

# References


[1] A. Z. Broder, The $r$-Stirling numbers, *Discrete Math.* **49** (1984), 241–259.

[2] A. Cayley, On the analytical forms called trees, *Amer. J. Math.* **4** (1881), 266–268.

[3] C. Chuan-Chong and K. Khee-Meng, *Principles and Techniques in Combinatorics*, World Scientific, 1992.

[4] I. Mező, Periodicity of the last digits of some combinatorial sequences, *J. Integer Sequences* **17** (2014), Article 14.1.1.

[5] I. Mező and J. L. Ramírez, Some identities of the $r$-Whitney numbers, *Aequationes Math.* **90** (2016), 393–406.

[6] B. Poonen, Periodicity of a combinatorial sequence, *Fibonacci Quart.* **26** (1988), 70–76.


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